On the Schrödinger Equation for the Supersymmetric FRW model

J.J. Rosales[*], **V.I. Tkach**[†]

Instituto de Física, Universidad de Guanajuato,
Apartado Postal E-143, C.P. 37150, León, Gto. México.

A.I. Pashnev[‡]

JINR-Bogoliubov Laboratory of Theoretical Physics
141980 Dubna, Moscow Region, Russia.

We consider a time-dependent Schrödinger equation for the Friedmann-Robertson-Walker (FRW) model. We show that for this purpose it is possible to include an additional action invariant under reparametrization of time. The last one does not change the equations of motion for the minisuperspace model, but changes only the constraint. The same procedure is applied to the supersymmetric case.

PACS numbers: 04.65.+e, 98.80Hw

[*]E-mail: juan@ifug3.ugto.mx

[†]E-mail: vladimir@ifug3.ugto.mx

[‡]E-mail: pashnev@thsun1.jinr.ru

I. INTRODUCTION

One of the most important questions in quantum cosmology is that of identifying a suitable time parameter [1] and a time-dependent Wheeler-DeWitt equation [2,3]. The main peculiarity of the gravity theory is the presence of non-physical variables (gauge variables) and constraints [3–6]. They arise due to the general coordinate invariance of the theory. The conventional Wheeler-DeWitt formulation gives a time independent quantum theory [7]. The canonical quantization of the minisuperspace approximation [8] has been used to find results in the hope, that they would illustrate the behaviour of general relativity [9]. In the minisuperspace models [2,7] there is a residual invariance under reparametrization of time (world-line symmetry). Due to this fact the equation that governs the quantum behaviour of these models is the Schrödinger equation for states with zero energy. On the other hand, supersymmetry transformations are more fundamental than time translations (reparametrization of time) in the sense, that these ones may be generated by anticommutators of the supersymmetry generators. The recent introduction of supersymmetric minisuperspace models has led to the square root equations for states with zero energy [10–12]. The structure of the world-line supersymmetry transformations has led to the zero Hamiltonian phenomena [2,6,12]. Investigations about time evolution problem for such quantum systems have been carried in two directions: the cosmological models of gravity have been quantized by reducing the phase space degrees of freedom [13–17] and with the help of the WKB approach [18–20].

In this work we consider a time-dependent Schrödinger equation for the homogeneous cosmological models. In our approach this equation arises due to an additional action invariant under time reparametrization. The last one does not change the equations of motion, but the constraint which becomes time-dependent Schrödinger equation. In the case of the supersymmetric minisuperspace model we obtain the supersymmetric constraints, one of them is a square root of time-dependent Schrödinger equation.

The paper is organized as follows: In section 2, we applied the canonical quantization procedure for the reparametrization invariant action. The extension to supersymmetric FRW model is performed in section 3.

II. REPARAMETRIZATION INVARIANCE

We begin by considering an homogeneous and isotropic metric defined by the line element

$$ds^2 = -N^2(t)dt^2 + R^2(t)d\Omega_3^2, \tag{1}$$

where the only dynamical degree of freedom is the scale factor $R(t)$. The lapse function $N(t)$, being a pure gauge variable, is not dynamical. The quantity $d\Omega_3^2$ is the standard line element on the unit three-sphere. We shall set $c = \hbar = 1$. The pure gravitational action corresponding to the metric (1) is

$$S_g = \frac{6}{\kappa^2}\int \left(-\frac{R\dot{R}^2}{2N} + \frac{1}{2}kNR\right) dt, \tag{2}$$

where $k = 1, 0, -1$ corresponds to a closed, flat or open space. $\kappa^2 = 8\pi G_N$, where G_N is the Newton's constant of gravity, and the overdot denotes differentiation with respect to t. The action (2) preserves the invariance under time reparametrization

$$t' \to t + a(t), \tag{3}$$

in this case, $N(t)$ and $R(t)$ transform as

$$\delta R = a\dot{R} \qquad\qquad \delta N = (aN)^{\cdot}, \tag{4}$$

that is $R(t)$ transforms as a scalar and $N(t)$ as a one-dimensional vector, and its dimensionality is the inverse of $a(t)$.

So, we consider the interacting action for the homogeneous real scalar matter field $\phi(t)$ and the scale factor $R(t)$. The action has the form

$$S_m = \int \left(\frac{R^3\dot{\phi}^2}{2N} - NR^3V(\phi) \right) dt, \tag{5}$$

this action is invariant under the local transformation (3), if in addition to law transformations for $R(t)$ and $N(t)$ in (4), the matter field transforms as a scalar $\delta\phi = a\dot{\phi}$. Thus, our system is described by the full action

$$S = S_g + S_m = \int \left(-\frac{3R\dot{R}^2}{\kappa^2 N} + \frac{R^3\dot{\phi}^2}{2N} + \frac{3kNR}{\kappa^2} - NR^3V(\phi) \right) dt. \tag{6}$$

Now, we shall consider the Hamiltonian analysis of this action. The canonical momenta for the variables R and ϕ are given by

$$P_R = \frac{\partial L}{\partial \dot{R}} = -\frac{6R\dot{R}}{\kappa^2 N}, \qquad P_\phi = \frac{R^3\dot{\phi}}{N}. \tag{7}$$

Their canonical Poisson brackets are defined as

$$\{R, P_R\} = 1, \qquad\qquad \{\phi, P_\phi\} = 1. \tag{8}$$

The canonical momentum for the variable $N(t)$ is

$$P_N \equiv \frac{\partial L}{\partial \dot{N}} = 0, \tag{9}$$

this equation merely constrains the variable $N(t)$. The canonical Hamiltonian can be calculated in the usual way, it has the form $H_c = NH_0$, then the total Hamiltonian is

$$H_T = NH_0 + u_N P_N, \tag{10}$$

where u_N is the Lagrange multiplier associated to the constraint $P_N = 0$ in (9), and H_0 is the Hamiltonian

$$H_0 = \left(-\frac{\kappa^2 P_R^2}{12R} + \frac{\pi_\phi^2}{2R^3} - \frac{3kR}{\kappa^2} + R^3V(\phi) \right). \tag{11}$$

The time evolution of any dynamical variables is generated by (10). For the compatibility of the constraint the Eq. (9) and the dynamics generated by the total Hamiltonian of Eq. (10), the following equation must hold

$$H_0 = 0, \tag{12}$$

which constrains the dynamics of our system. So, we proceed to the quantum mechanics from the above classical system. We introduce the wave function of the Universe ψ. The constraint equation (12) must be imposed on the states

$$H_0 \psi = 0. \tag{13}$$

This constraint nullifies all the dynamical evolution generated by the total Hamiltonian (10). A commutator of any operator and the total Hamiltonian becomes zero, if it is evaluated for the above constrained states.The disappearence of time seems disappointing, however, it is a proper consequence of the invariance of general coordinate transformation in general relativity. The equation (9) merely says, that the wave function ψ does not depend on the lapse function $N(t)$. Therefore, we expect that the equation in (13) may contain any information of dynamics, since the WKB solutions of the equation (13) is indeed parametrized by an "external" time [20]. In the WKB approach the coordinate T is usually called WKB time. In quantum cosmology the constraint (13) is known as the Wheeler-DeWitt equation (time-independent Schrödinger equation).

In order to get a time-dependent Schrödinger equation we introduce the time parameter $T(t)$ using the relation $N(t) = \frac{dT}{dt}$. We shall regard the following invariant action

$$S_r = \frac{1}{\kappa^3} \int R^3 P_T \left(\frac{dT(t)}{dt} - N(t) \right) dt, \tag{14}$$

where (T, P_T) is a pair of canonical variables, P_T is the momentum conjugate to T. This action is invariant under reparametrization of time (3), if P_T and T transform as a scalars under reparametrization (3)

$$\delta P_T = a(t) \dot{P}_T \qquad\qquad \delta T = a(t) \dot{T}, \tag{15}$$

and N, R as in (4).

So, adding the action (14) to the action (6) we have the total invariant action $\tilde{S} = S_g + S_m + S_r$. In the first order form, we get

$$\tilde{S} = \int \left\{ \dot{R} P_R + \dot{\phi} P_\phi - N H_0 + \frac{R^3 P_T}{\kappa^3} \left(\frac{dT}{dt} - N(t) \right) \right\} dt. \tag{16}$$

We shall proceed with the canonical quantization of the action (16). We define the canonical momenta π_T and π_{P_T} corresponding to the variables T and P_T, respectively. We get

$$\pi_T \equiv \frac{\partial \tilde{L}}{\partial \dot{T}} = \frac{R^3}{\kappa^3} P_T, \qquad\qquad \pi_{P_T} \equiv \frac{\partial \tilde{L}}{\partial \dot{P}_T} = 0, \tag{17}$$

leading to the constraints

$$\Pi_1 \equiv \pi_T - \frac{R^3}{\kappa^3} P_T = 0, \qquad \Pi_2 \equiv \pi_{P_T} = 0. \tag{18}$$

So, we define the matrix C_{AB}, $(A, B = 1, 2)$ as a Poisson brackets between the constraints $C_{AB} = \{\Pi_A, \Pi_B\}$. Then, we have the following non-zero matrix elements

$$\{\Pi_1, \Pi_2\} = -\frac{R^3}{\kappa^3}, \tag{19}$$

with their inverse matrix elements $(C^{-1})^{1,2} = -\frac{\kappa^3}{R^3}$. The Dirac's brackets $\{,\}^*$ are defined by

$$\{f, g\}^* = \{f, g\} - \{f, \sqcap_A\} C^{-1}_{AB} \{\sqcap_B, g\}. \tag{20}$$

The result of this procedure leads to the non-zero Dirac's brackets

$$\{T, P_T\}^* = \frac{\kappa^3}{R^3}. \tag{21}$$

Then, the canonical Hamiltonian is

$$\tilde{H}_c = N\left(\frac{R^3}{\kappa^3} P_T + H_0\right), \tag{22}$$

where the Hamiltonian constraint corresponding to the action (16) is

$$\tilde{H} = \frac{R^3}{\kappa^3} P_T + H_0. \tag{23}$$

At the quantum level the Dirac's brackets become commutators

$$[T, P_T] = i\{T, P_T\}^* = i\frac{\kappa^3}{R^3}. \tag{24}$$

So, taking the momentum P_T corresponding to T as

$$P_T = -i\frac{\kappa^3}{R^3}\frac{\partial}{\partial T}, \tag{25}$$

the constraint (23) becomes quantum condition on the wave function ψ,

$$i\frac{\partial}{\partial T}\psi(T, R, \phi) = H\left(-i\frac{\partial}{\partial R}, -i\frac{\partial}{\partial \phi}, R, \phi\right)\psi. \tag{26}$$

There is a question of the factor ordering in the differential equation (26). In order to find a correct quantum expression for the Hamiltonian, we must always consider the factor ordering ambiguites

$$i\frac{\partial}{\partial T}\psi = \left[\frac{\kappa^2}{12} R^{-p-1}\frac{\partial}{\partial R} R^p \frac{\partial}{\partial R} - \frac{1}{2R^3}\frac{\partial^2}{\partial \varphi^2} - \frac{3kR}{\kappa^2} + R^3 V(\varphi)\right]\psi. \tag{27}$$

The parameter p takes into account some of the factor ordering ambiguity of the theory. The equation (26) is the time-dependent Schrödinger equation for minisuperspace.

The equations of motion are obtained by demanding that the action $\tilde{S} = S_g + S_m + S_r$ is extremal, *i.e.* the functional derivatives of $\tilde{S}$ must be zero

$$\frac{\delta \tilde{S}}{\delta R} = \frac{\delta S_g}{\delta R} + \frac{\delta S_m}{\delta R} + \frac{\delta S_r}{\delta R} = 0. \tag{28}$$

As a consequence of the equation of motion

$$\frac{\delta \tilde{S}}{\delta P_T} = \frac{\delta S_r}{\delta P_T} = \frac{R^3}{\kappa^3}(\dot{T} - N) = 0, \tag{29}$$

the last term in (28) $\frac{3R^2}{\kappa^3} P_T(\dot{T} - N)$ dissapears and, in fact, inclusion in S of an additional invariant action S_r does not change the equations of motion except the equation $\frac{\delta \tilde{S}}{\delta N} = 0$, which is the constraint (23).

In the case of general relativity, the canonical quantization makes use of the $3+1$ spliting of the space-time geometry by Arnowit-Deser-Misner (ADM) [3]. Acording to the ADM prescription of general relativity one consider a slicing of the space-time by a family of space-like hypersurfaces labeled by t. This can be thought of as a time coordinate, so that each slice is identified by the relation $t = const$. So, we introduce a time $T(t, x)$, which will be related to time t as

$$n^\mu \partial_\mu T = 1. \tag{30}$$

We consider time T as a canonical coordinate. Its corresponding canonical conjugate momentum will be P_T. These T and P_T transform under general coordinate transformations as scalars. So, in this case, the additional action term, which is invariant under general coordinate transformations can be written in the following form

$$\begin{aligned} S_{(d=4)} &= \frac{1}{\kappa^3} \int \sqrt{-g} P_T(n^\mu \partial_\mu T - 1) d^4x \\ &= \frac{1}{\kappa^3} \int N h^{1/2} P_T\Big(\frac{\partial_0 T}{N} - \frac{N^i \partial_i T}{N} - 1\Big) dt d^3x \\ &= \frac{1}{\kappa^3} \int h^{1/2} P_T(\partial_0 T - N^i \partial_i T - N) dt d^3x, \end{aligned} \tag{31}$$

varying this action with respect to the three metric h_{ik} and P_T, we get

$$\begin{aligned} \frac{\delta S_{(d=4)}}{\delta h_{ik}} &= \frac{1}{2} \frac{h^{1/2}}{\kappa^3} h^{ik} P_T(\partial_0 T - N^i \partial_i T - N) = 0 \\ \frac{\delta S_{(d=4)}}{\delta P_T} &= \frac{h^{1/2}}{\kappa^3}(\partial_0 T - N^i \partial_i T - N) = 0. \end{aligned} \tag{32}$$

So, given a four dimensional space-time geometry with the metric $g_{\mu\nu}$ considered as a parameter family of three-dimensional space-like hypersurfaces $t = const$ the intrinsic metric of each surface is $h_{ij} = g_{ij}$, $g = \det g_{\mu\nu}$, $h = \det h_{ij}$ and $g = Nh$. The unit future-directed normal vector is n^μ, $(n^\mu n_\mu = -1)$ to hypersurface $t = const$ with component $n_\mu = (-N, 0, 0, 0)$

and $n^\mu = (\frac{1}{N}, -\frac{N^i}{N})$, where N^i is the shift vector, (for the metric (1) the shift vector is $N^i = 0$) and $h^{1/2} = R^3$.

So, if we consider the four dimensional gravity interacting with a scalar matter field and the invariant additional term (31) after applying the ADM spliting $(3+1)$ formalism for the FRW model, we get

$$
\begin{aligned}
S = &-\frac{1}{2\kappa^2}\int \sqrt{-g}Rd^4x - \int \sqrt{-g}\Big[\frac{(\partial_\mu\phi)^2}{2} + V(\phi)\Big]d^4x \\
&+ \frac{1}{\kappa^3}\int \sqrt{-g}P_T(n^\mu\partial_\mu T - 1)d^4x = \int\Big[\Big(-\frac{3R\dot{R}^2}{2N\kappa^2} + \frac{3}{\kappa^2}kNR\Big) \\
&+ \frac{R^3\dot{\phi}^2}{2N} - NR^3V(\phi)\Big]dt + \frac{1}{\kappa^3}\int R^3 P_T\Big(\frac{dT(t)}{dt} - N(t)\Big)dt.
\end{aligned}
\tag{33}
$$

In particular, putting the gauge $N = 1$, then $T = t$ and we obtain the so-called cosmic time, on the other hand, if we take $N = \frac{R}{\kappa}$ then we get the conformal time gauge. In terms of the $(3+1)$ variables, the action (33) takes the form [20]

$$
\begin{aligned}
S = &\frac{1}{2\kappa^2}\int Nh^{1/2}\Big(K_{ij}K^{ij} - K^2 +^{(3)} R\Big)dtd^3x \\
&+\frac{1}{\kappa^3}\int h^{1/2}P_T\Big(\partial_0 T - N^i\partial_i T - N\Big)dtd^3x + S_{matter},
\end{aligned}
\tag{34}
$$

where K is the trace of the extrinsic curvature K_{ij}. In the action (34) we have ignored the surface term. Then, the canonical Hamiltonian is

$$
\begin{aligned}
\tilde{H} = &N\Big(-\frac{h^{1/2}}{\kappa^3}P_T + \frac{\kappa^2}{2}G_{ijkl}\pi^{ij}\pi^{kl} - \frac{1}{2\kappa^2}(h^{1/2})^{(3)}R + H_{matter}(\phi, \pi_\phi)\Big) \\
&+N^i\Big(\frac{1}{\kappa^3}P_T\partial_i T - 2D_j\pi^j_i + H_{i(matter)}\Big),
\end{aligned}
\tag{35}
$$

where π^{ij} and π_ϕ are the momenta conjugated to h_{ij} and ϕ, respectively. D_i is a covariant derivative on the metric h_{ij} and $G_{ijkl} = \frac{1}{2}h^{-1/2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})$. The Dirac quantization of this model will lead to the many-fingered time Schrödinger equation (Tomanaga-Shwinger equation) for the wave function $\Psi(T, h_{ik}, \phi)$

$$
\begin{aligned}
i\frac{\delta\Psi}{\delta T} = &\Big(-\frac{1}{2\kappa}G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} + \frac{1}{2\kappa}b_{ik}\frac{\delta}{\delta h_{ik}} - \frac{1}{2\kappa^3}h^{-1/2}\frac{\delta^2}{\delta\phi^2} \\
&-\frac{\kappa}{2}(h^{1/2})^{(3)}R + \frac{\kappa^3}{2}h^{1/2}h^{ik}\phi_{,i}\phi_{,k} + \kappa^3 V(\phi)\Big),
\end{aligned}
\tag{36}
$$

where the coefficients b_{ik} depend of the chose of factor ordering in the term $-\frac{1}{2\kappa}G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}}$.

III. SUPERSYMMETRIC FRW MODEL

In order to obtain a superfield formulation of the action (6) the transformation of the time reparametrization (3) must be extended to the $n = 2$ local conformal time supersymmetry (LCTS) $(t, \eta, \bar{\eta})$ [21,22]. These transformations can be written as

$$\delta t = \mathbb{L}(t,\theta,\bar\theta) + \frac{1}{2}\bar\theta D_{\bar\theta}\mathbb{L}(t,\theta,\bar\theta) - \frac{1}{2}\theta D_\theta \mathbb{L}(t,\theta,\bar\theta),$$
$$\delta\theta = \frac{i}{2}D_\theta \mathbb{L}(t,\theta,\bar\theta), \qquad \delta\bar\theta = \frac{i}{2}D_{\bar\theta}\mathbb{L}(t,\theta,\bar\theta), \tag{37}$$

with the superfunction $\mathbb{L}(t,\theta,\bar\theta)$ defined by

$$\mathbb{L}(t,\theta,\bar\theta) = a(t) + i\theta\bar\beta'(t) + i\bar\theta\beta'(t) + b(t)\theta\bar\theta, \tag{38}$$

where $D_\theta = \frac{\partial}{\partial\theta} + i\bar\theta\frac{\partial}{\partial t}$ and $D_{\bar\theta} = -\frac{\partial}{\partial\bar\theta} - i\theta\frac{\partial}{\partial t}$ are the supercovariant derivatives of the $n=2$ supersymmetry, $a(t)$ is a local time reparametrization parameter, $\beta'(t)$ is the Grassmann complex parameter of the local conformal $n=2$ supersymmetry transformations and $b(t)$ is the parameter of the local $U(1)$ rotations on the Grassmann coordinates θ ($\bar\theta = \theta^\dagger$). Then, the superfield generalization of the action (6), which is invariant under the $n=2$ (LCTS) (37) has the form [23,24]

$$S_{(n=2)} = S_g + S_m = \int \left(-\frac{3}{\kappa^2}\mathbb{N}^{-1}\mathbb{R}D_{\bar\theta}\mathbb{R}D_\theta\mathbb{R} + \frac{3\sqrt{k}}{\kappa^2}\mathbb{R}^2\right) d\theta d\bar\theta dt \tag{39}$$
$$+ \int \left(\frac{1}{2}\mathbb{N}^{-1}\mathbb{R}^3 D_{\bar\theta}\mathbf{\Phi} D_\theta\mathbf{\Phi} - 2\mathbb{R}^3 g(\mathbf{\Phi})\right) d\theta d\bar\theta dt,$$

where $g(\Phi)$ is the superpotential. The most general supersymmetric interaction for a set of complex homogeneous scalar fields with the scale factor was considered in [25,26]. For the one-dimensional gravity superfield $\mathbb{N}(t,\theta,\bar\theta)$ we have the following series expansion

$$\mathbb{N}(t,\theta,\bar\theta) = N(t) + i\theta\bar\psi'(t) + i\bar\theta\psi'(t) + V'(t)\theta\bar\theta, \tag{40}$$

where $N(t)$ is the lapse function, $\psi' = N^{1/2}\psi$ and $V' = NV + \bar\psi\psi$. The components $N, \psi, \bar\psi$ and V in (40) are gauge fields of the one-dimensional $n=2$ supergravity. The superfield (40) transforms as one-dimensional vector under the (LCTS) (37),

$$\delta\mathbb{N} = (\mathbb{L}\mathbb{N})^{\cdot} + \frac{i}{2}D_{\bar\theta}\mathbb{L}D_\theta\mathbb{N} + \frac{i}{2}D_\theta\mathbb{L}D_{\bar\theta}\mathbb{N}. \tag{41}$$

The series expansion for the superfield $\mathbb{R}(t,\theta,\bar\theta)$ has the form

$$\mathbb{R}(t,\theta,\bar\theta) = R(t) + i\theta\bar\lambda'(t) + i\bar\theta\lambda'(t) + B'(t)\theta\bar\theta, \tag{42}$$

where $R(t)$ is the scale factor of the FRW Universe, $\lambda' = \kappa N^{1/2}\lambda$ and $B' = \kappa NB + \frac{\kappa}{2}(\bar\psi\lambda - \psi\bar\lambda)$.

For the real scalar matter superfield $\mathbf{\Phi}(t,\theta,\bar\theta)$ we have

$$\mathbf{\Phi}(t,\theta,\bar\theta) = \phi(t) + i\theta\bar\chi'(t) + i\bar\theta\chi'(t) + F'(t)\theta\bar\theta, \tag{43}$$

where $\chi' = N^{1/2}\chi$ and $F' = NF + \frac{1}{2}(\bar\psi\bar\chi - \psi\chi)$. The components $B(t)$ and $F(t)$ in the superfields $\mathbb{R}$ and $\mathbf{\Phi}$ are auxiliary fields. The superfields (42) and (43) transform as scalars under the transformations (37).

Performing the integration over $\theta, \bar\theta$ in (39) and eliminating the auxiliary fields B and F by means of their equations of motion, the action (39) takes its component form. The first-class constraints may be obtained from the component form of the action (39) varying it with respect to $N(t), \psi(t), \bar{\psi}(t)$ and $V(t)$, respectively. Then, we obtain the following first-class constraints $H_0 = 0$, $S = 0$, $\bar{S} = 0$ and $F = 0$, where

$$\begin{aligned} H_0 = & -\frac{\kappa^2}{12}\frac{\pi_R^2}{R} - \frac{3kR}{\kappa^2} - \frac{\sqrt{k}}{3R}\bar{\lambda}\lambda + \frac{\pi_\phi^2}{2R^3} - \frac{i\kappa}{2R^3}\pi_\phi(\bar{\lambda}\chi + \lambda\bar{\chi}) - \frac{\kappa^2}{4R^3}\bar{\lambda}\lambda\bar{\chi}\chi \\ & + \frac{3\sqrt{k}}{2R}\bar{\chi}\chi + \kappa^2 g(\phi)\bar{\lambda}\lambda + 6\sqrt{k}g(\phi)R^2 + 2\left(\frac{\partial g}{\partial\phi}\right)^2 R^3 - 3\kappa^2 g^2(\phi)R^3 \\ & + \frac{3}{2}\kappa^2 g(\phi)\bar{\chi}\chi + 2\frac{\partial^2 g}{\partial\phi^2}\bar{\chi}\chi + \kappa\frac{\partial g}{\partial\phi}(\bar{\lambda}\chi - \lambda\bar{\chi}), \end{aligned} \tag{44}$$

$$\begin{aligned} S = & \left(\frac{i\kappa}{3}R^{-1/2}\pi_R - \frac{2\sqrt{k}}{\kappa}R^{1/2} + 2\kappa g(\phi)R^{3/2} + \frac{\kappa}{4}R^{-3/2}\bar{\chi}\chi\right)\lambda \\ & + \left(iR^{-3/2}\pi_\phi + 2R^{3/2}\frac{\partial g}{\partial\phi}\right)\chi, \end{aligned} \tag{45}$$

and

$$F = -\frac{2}{3}\bar{\lambda}\lambda + \bar{\chi}\chi, \tag{46}$$

where $\bar{S} = S^\dagger$.

The canonical Hamiltonian is the sum of all the constraints

$$H_{c(n=2)} = NH_0 + \frac{1}{2}\bar{\psi}S - \frac{1}{2}\psi\bar{S} + \frac{1}{2}VF. \tag{47}$$

In terms of Dirac's brackets for the canonical variables $R, \pi_R, \phi, \pi_\phi, \lambda, \bar{\lambda}, \chi$ and $\bar{\chi}$ the quantities $H_0, S, \bar{S}$ and F form the closed super-algebra of conserving charges

$$\begin{aligned} &\{S, \bar{S}\}^* = -2iH_0, \qquad [H_0, S]^* = [H_0, \bar{S}]^* = 0 \\ &[F, S]^* = iS, \qquad [F, \bar{S}]^* = -i\bar{S}. \end{aligned} \tag{48}$$

So, any physically allowed states must obey the following quantum constraints

$$H_0\psi = 0, \qquad S\psi = 0, \qquad \bar{S}\psi = 0, \qquad F\psi = 0, \tag{49}$$

when we change the classical variables by their corresponding operators. The first equation in (49) is the Wheeler-DeWitt equation for the minisuperspace model. Therefore, we have the time-independent Schrödinger equation, this fact is due to the invariance under reparametrization symmetry of the action (39), this problem is well-known as the "problem of time" [1] in the minisuperspace models and general relativity theory. Due to the super-algebra (48) the second and the thirth equations in (49) reflect the fact, that there is a "square root" of the Hamiltonian H_0 with zero energy states. The constraints Hamiltonian H_0, supercharges $S, \bar{S}$, F follow from the invariance of the action (39) under the $n = 2$ (LCTS) transformations (37).

In order to have a time-dependent Schrödinger equation for the supersymmetric minisuperspace models with the action (39) we consider a generalization of the reparametrization invariant action S_r (14). In the case of $n = 2$ (LCTS) it has the superfield form

$$S_{r(n=2)} = -\int \left[I\!\!P - \frac{i}{2} I\!\!N^{-1} \left(D_{\bar\theta} \mathbf{T} D_\theta I\!\!P - D_{\bar\theta} I\!\!P D_\theta \mathbf{T} \right) \right] d\theta d\bar\theta dt. \tag{50}$$

Note, that the $BerE_B^A$, as well as the superjacobian of $n = 2$ (LCTS) transformations, is equal to one and is omitted in the actions (39,50). The action (50) is determined in terms of the new superfields $\mathbf{T}$ and $I\!\!P$. The series expansion for $\mathbf{T}$ has the form

$$\mathbf{T}(t,\theta,\bar\theta) = T(t) + \theta\eta'(t) - \bar\theta\bar\eta'(t) + m'(t)\theta\bar\theta, \tag{51}$$

where $\eta' = N^{1/2}\eta$ and $m' = Nm + \frac{i}{2}(\bar\psi\bar\eta + \psi\eta)$. The superfield $\mathbf{T}$ is determined by the odd complex time variables $\eta(t)$ and $\bar\eta(t)$, which are the superpartners of the time $T(t)$ and one auxiliary parameter $m(t)$.

The transformation rule for the superfield $\mathbf{T}(t,\theta,\bar\theta)$ under the $n = 2$ (LCTS) transformations is

$$\delta\mathbf{T} = I\!\!L\dot{\mathbf{T}} + \frac{i}{2} D_{\bar\theta} I\!\!L D_\theta \mathbf{T} + \frac{i}{2} D_\theta I\!\!L D_{\bar\theta} \mathbf{T}, \tag{52}$$

and transforms as a scalar under the transformations (37). The superfield $I\!\!P(t,\theta,\bar\theta)$ has the form

$$I\!\!P(t,\theta,\bar\theta) = \rho(t) + i\theta P'_{\bar\eta}(t) + i\bar\theta P'_\eta(t) + P_T(t)\theta\bar\theta, \tag{53}$$

where $P'_\eta = N^{1/2}P_\eta$ and $P'_T = NP_T + \frac{1}{2}(\bar\psi P_\eta - \psi P_{\bar\eta})$, P_η and $P_{\bar\eta}$ are the odd complex momenta, *i.e.* the superpartners of the momentum P_T.

The superfield $I\!\!P(t,\theta,\bar\theta)$ transforms as

$$\delta I\!\!P(t,\theta,\bar\theta) = I\!\!L\dot{I\!\!P} + \frac{i}{2} D_{\bar\theta} I\!\!L D_\theta I\!\!P + \frac{i}{2} D_\theta I\!\!L D_{\bar\theta} I\!\!P. \tag{54}$$

The action (50) is invariant under the $n = 2$ (LCTS) transformtions (37). Performing the integration over θ and $\bar\theta$ in (50) and making the redefinitions $P_T \to \frac{R^3}{\kappa^3}P_T$, $P_\eta \to \frac{R^3}{\kappa^3}P_\eta$ and $P_{\bar\eta} \to \frac{R^3}{\kappa^3}P_{\bar\eta}$ we obtain the component action

$$\begin{aligned} S_{r(n=2)} = -\int \Bigg\{ \frac{R^3}{\kappa^3} \Bigg(& P_T(N - \dot T) + i\dot\eta P_\eta + i\dot{\bar\eta} P_{\bar\eta} + \frac{\bar\psi}{2}(P_\eta - \bar\eta P_T) \\ - & \frac{\psi}{2}(P_{\bar\eta} - \eta P_T) + \frac{V}{2}(\eta P_\eta - \bar\eta P_{\bar\eta}) \Bigg) + m\dot\rho - \frac{i}{2} m\psi P_{\bar\eta} - \frac{i}{2} m\bar\psi P_\eta \Bigg\} dt. \end{aligned} \tag{55}$$

We can see from (55) that the momenta $P_\eta, P_{\bar\eta}$ and P_T in the superfield (53) are related with the components of the superfield (40), which enter in the action (39), unlike those momenta, the component ρ of the superfield (53) and the component of the superfield (51) are not related with any components in (40). Hence, one can show that the variables ρ and m are auxiliary fields in the sense, that they can be eliminated from (55) by some unitary transformation after this, we have

$$\begin{aligned} S_{r(n=2)} = -\int \frac{R^3}{\kappa^3} \Bigg\{ & P_T(N - \dot T) + i\dot\eta P_\eta + i\dot{\bar\eta} P_{\bar\eta} + \frac{\bar\psi}{2}(P_\eta - \bar\eta P_T) \\ - & \frac{\psi}{2}(P_{\bar\eta} - \eta P_T) + \frac{V}{2}(\eta P_\eta - \bar\eta P_{\bar\eta}) \Bigg\} dt. \end{aligned} \tag{56}$$

In addition to the canonical momenta π_T and π_{P_T} for the two even constraints (17), the action (56) has the additional momenta $\mathcal{P}_\eta$ and $\mathcal{P}_{P_\eta}$ conjugate to η and P_η, respectively,

$$\mathcal{P}_\eta = \frac{\partial L_{r(n=2)}}{\partial \dot{\eta}} = -i\frac{R^3}{\kappa^3}P_\eta, \qquad \mathcal{P}_{P_\eta} = \frac{\partial L_{r(n=2)}}{\partial \dot{P}_\eta} = 0. \tag{57}$$

With respect to the canonical odd Poisson brackets we have

$$\{\eta, \mathcal{P}_\eta\} = 1, \qquad \{P_\eta, \mathcal{P}_{P_\eta}\} = 1. \tag{58}$$

They form two primary constraints of the second class

$$\sqcap_3(\eta) \equiv \mathcal{P}_\eta + i\frac{R^3}{\kappa^3}P_\eta = 0, \qquad \sqcap_4(P_\eta) \equiv \mathcal{P}_{P_\eta} = 0. \tag{59}$$

The only non-vanishing Poisson bracket between these constraints is

$$\{\sqcap_3, \sqcap_4\} = i\frac{R^3}{\kappa^3}. \tag{60}$$

The momenta $\mathcal{P}_{\bar{\eta}}$ and $\mathcal{P}_{P_{\bar{\eta}}}$ conjugate to $\bar{\eta}$ and $P_{\bar{\eta}}$ respectively, it also gives two primary constraints of the second-class

$$\sqcap_5(\bar{\eta}) \equiv \mathcal{P}_{\bar{\eta}} + i\frac{R^3}{\kappa^3}P_{\bar{\eta}} = 0, \qquad \sqcap_6(P_{\bar{\eta}}) = \mathcal{P}_{P_{\bar{\eta}}} = 0, \tag{61}$$

with non-vanishing Poisson bracket

$$\{\sqcap_5, \sqcap_6\} = i\frac{R^3}{\kappa^3}. \tag{62}$$

The constraints (59) and (61) for the Grassmann dynamical variables can be eliminated by Dirac's procedure. Defining the matrix constraint $C_{ik}(i,k = \eta, P_\eta, \bar{\eta}, P_{\bar{\eta}})$ as the odd Poisson bracket we have the following non-zero matrix elements

$$\begin{aligned} C_{\eta P_\eta} &= C_{P_\eta \eta} = \{\sqcap_3, \sqcap_4\} = i\frac{R^3}{\kappa^3}, \\ C_{\bar{\eta} P_{\bar{\eta}}} &= C_{P_{\bar{\eta}} \bar{\eta}} = \{\sqcap_5, \sqcap_6\} = i\frac{R^3}{\kappa^3}, \end{aligned} \tag{63}$$

with their inverse matrices $(C^{-1})^{\eta P_\eta} = -i\frac{\kappa^3}{R^3}$ and $(C^{-1})^{\bar{\eta} P_\eta} = -i\frac{\kappa^3}{R^3}$. The result of this procedure is the elimination of the momenta conjugate to the Grassmann variables, leaving us with the following non-zero Dirac's bracket relations

$$\{\eta, P_\eta\}^* = i\frac{\kappa^3}{R^3}, \qquad \{\bar{\eta}, P_{\bar{\eta}}\}^* = i\frac{\kappa^3}{R^3}. \tag{64}$$

So, if we take the additional term (50), then the full action is

$$\tilde{S}_{(n=2)} = S_{(n=2)} + S_{r(n=2)}. \tag{65}$$

The canonical Hamiltonian for the action (65) will have the following form

$$\tilde{H}_{c(n=2)} = N\left(\frac{R^3}{\kappa^3}P_T + H_0\right) + \frac{\bar{\psi}}{2}\left(-\frac{R^3}{\kappa^3}S_\eta + S\right)$$
$$- \frac{\psi}{2}\left(\frac{R^3}{\kappa^3}S_{\bar{\eta}} + \bar{S}\right) + \frac{V}{2}\left(\frac{R^3}{\kappa^3}F_\eta + F\right), \qquad (66)$$

where $S_\eta = (-P_\eta + \eta P_T)$, $S_{\bar{\eta}} = (P_{\bar{\eta}} - \eta P_T)$, $F_\eta = (\eta P_\eta - \bar{\eta} P_{\bar{\eta}})$, and $H_0, S, \bar{S}$ and F are defined in (44,45,46). In the component form of the action (65) there are no kinetic terms for $N, \psi, \bar{\psi}$ and V. This fact is reflected in the primary constraints $P_N = 0$, $P_\psi = 0$, $P_{\bar{\psi}} = 0$ and $P_V = 0$, where $P_N, P_\psi, P_{\bar{\psi}}$ and P_V are the canonical momenta conjugated to $N, \psi, \bar{\psi}$ and V, respectively. Then, the total Hamiltonian may be written as

$$\tilde{H} = \tilde{H}_{c(n=2)} + u_N P_N + u_\psi P_\psi + u_{\bar{\psi}} P_{\bar{\psi}} + u_V P_V. \qquad (67)$$

Due to the conditions $\dot{P}_N = \dot{P}_\psi = \dot{P}_{\bar{\psi}} = \dot{P}_V = 0$ we now have the first-class constraints

$$\tilde{H} = \frac{R^3}{\kappa^3}P_T + H_0 = 0, \qquad \mathcal{F} = \frac{R^3}{\kappa^3}F_\eta + F = 0,$$
$$Q_\eta = -\frac{R^3}{\kappa^3}S_\eta + S = 0, \qquad Q_{\bar{\eta}} = \frac{R^3}{\kappa^3}S_{\bar{\eta}} + \bar{S} = 0. \qquad (68)$$

They form a closed super-algebra with respect to the Dirac's brackets

$$\{Q_\eta, Q_{\bar{\eta}}\}^* = -2i\tilde{H}, \qquad [\tilde{H}, Q_\eta]^* = [\tilde{H}, Q_{\bar{\eta}}]^* = 0$$
$$[\mathcal{F}, Q_\eta]^* = iQ_\eta, \qquad [\mathcal{F}, Q_{\bar{\eta}}]^* = -iQ_{\bar{\eta}}. \qquad (69)$$

After quantization Dirac's brackets must be replaced by anticommutators

$$\{\eta, P_\eta\} = i\{\eta, P_\eta\}^* = -\frac{\kappa^3}{R^3}, \qquad \{\bar{\eta}, P_{\bar{\eta}}\} = i\{\bar{\eta}, P_{\bar{\eta}}\}^* = -\frac{\kappa^3}{R^3}, \qquad (70)$$

with the operator relations

$$P_\eta = -\frac{\kappa^3}{R^3}\frac{\partial}{\partial \eta}, \qquad P_{\bar{\eta}} = -\frac{\kappa^3}{R^3}\frac{\partial}{\partial \bar{\eta}}. \qquad (71)$$

To obtain the quantum expression for $H_0, S, \bar{S}, F$ we must solve the operator ordering ambiguity. Such ambiguities always take place when the operator expression contains the product of non-commuting operators λ and $\bar{\lambda}, \chi$ and $\bar{\chi}$ R and $\pi_{\bar{R}} = -i\frac{\partial}{\partial R}, \phi$ and $\pi_\phi = -i\frac{\partial}{\partial \phi}$, such procedure leads in our case to the following expressions for the generators on the quantum level

$$\tilde{H} = -i\frac{\partial}{\partial T} + H_0(R, \pi_R, \phi, \pi_\phi, \lambda, \bar{\lambda}, \chi\bar{\chi}),$$
$$Q_\eta = -\Big(\frac{\partial}{\partial \eta} - i\bar{\eta}\frac{\partial}{\partial T}\Big) + S(R, \pi_R, \phi, \pi_\phi, \lambda, \chi), \qquad (72)$$
$$Q_{\bar{\eta}} = \Big(-\frac{\partial}{\partial \bar{\eta}} + i\eta\frac{\partial}{\partial T}\Big) + \bar{S}(R, \pi_R, \phi, \pi_\phi, \bar{\lambda}, \bar{\chi}),$$
$$\mathcal{F} = \Big(-\eta\frac{\partial}{\partial \eta} + \bar{\eta}\frac{\partial}{\partial \bar{\eta}}\Big) + F(\lambda, \bar{\lambda}, \chi, \bar{\chi}),$$

where $S_\eta = \frac{\partial}{\partial\eta} - i\bar\eta\frac{\partial}{\partial T}$ and $S_{\bar\eta} = -\frac{\partial}{\partial\bar\eta} + i\eta\frac{\partial}{\partial T}$ are the generators of the supertranslation, $P_T = -i\frac{\partial}{\partial T}$ is the ordinary time translation on the superspace with coordinates $(t, \eta, \bar\eta)$

$$\{S_\eta, S_{\bar\eta}\} = 2i\frac{\partial}{\partial T}, \tag{73}$$

and $F_\eta = -\eta\frac{\partial}{\partial\eta} + \bar\eta\frac{\partial}{\partial\bar\eta}$ is the $\mathcal{U}(1)$ generator of the rotation on the complex Grassmann coordinate $\eta(\bar\eta = \eta^\dagger)$. The algebra of the quantum generators of the conserving charges $H_0, S, \bar S, F$ is a closed super-algebra

$$\begin{aligned} \{S, \bar S\} = 2H_0, \quad [S, H_0] = [S, H_0] = [\tilde S, H_0] = [F, H_0] = 0 \\ S^2 = \bar S^2 = 0, \qquad [F, S] = -\bar S, \qquad [F, \bar S] = \bar S. \end{aligned} \tag{74}$$

We can see from Eqs. (69) and (72) that the operators $\tilde H, Q_\eta, Q_{\bar\eta}$ and $\mathcal{F}$ obey the same super-algebra (74)

$$\begin{aligned} \{Q_\eta, Q_{\bar\eta}\} = 2\tilde H, \quad [Q_\eta, \tilde H] = [Q_\eta, \tilde H] = [\mathcal{F}, \tilde H] = 0 \\ Q_\eta^2 = Q_{\bar\eta}^2 = 0, \qquad [\mathcal{F}, Q_\eta] = -Q_\eta, \quad [\mathcal{F}, Q_{\bar\eta}] = Q_{\bar\eta}. \end{aligned} \tag{75}$$

In the quantum theory the first-class constraints (72) become conditions on the wave function Ψ, which has the superfield form

$$\begin{aligned} \Psi(T, \eta, \bar\eta, R, \phi, \bar\phi, \lambda, \bar\lambda, \chi, \bar\chi) = \psi(T, R, \phi, \lambda, \bar\lambda, \chi, \bar\chi) \\ + i\eta\xi(T, R, \phi, \lambda, \bar\lambda, \chi, \bar\chi) + i\bar\eta\zeta(T, R, \phi, \lambda, \bar\lambda, \chi, \bar\chi) \\ + \sigma(T, R, \phi, \lambda, \bar\lambda, \chi, \bar\chi)\eta\bar\eta. \end{aligned} \tag{76}$$

So, we have the supersymmetric quantum constraints

$$\tilde H\Psi = 0, \qquad Q_\eta\Psi = 0, \qquad Q_{\bar\eta}\Psi = 0, \qquad \mathcal{F}\Psi = 0. \tag{77}$$

Taking the constraints

$$Q_\eta\Psi = 0, \qquad Q_{\bar\eta}\Psi = 0, \tag{78}$$

and due to the algebra (75)

$$\{Q_\eta, Q_{\bar\eta}\}\Psi = 2\tilde H\Psi = 0. \tag{79}$$

This is a time-dependent Schrödinger equation for the minisuperspace model.

The condition (79) leads to the following form for the wave function (76)

$$\psi_* = \psi + \eta(S\psi) + \bar\eta(\bar S\psi) - \frac{1}{2}(\bar S S - S\bar S)\psi\eta\bar\eta, \tag{80}$$

then $Q_\eta\psi_*$ has the following form

$$\begin{aligned} Q_\eta\psi_* = \bar\eta\Big(i\frac{d\psi}{dT} - \frac{1}{2}\{S, \bar S\}\psi\Big) + \\ + \eta\bar\eta S\Big(i\frac{d\psi}{dT} - \frac{1}{2}\{S, \bar S\}\psi\Big) = 0, \end{aligned} \tag{81}$$

this is the standard Schrödinger equation and due to the relation $H_0 = \frac{1}{2}\{S, \bar{S}\}$ it may be written as

$$i\frac{\partial \psi}{\partial T} = H_0 \psi, \tag{82}$$

where the wave function is $\psi(T, R, \lambda, \bar{\lambda}, \chi, \bar{\chi})$. If we put in the Schrödinger equation (82) the condition of a stationary state given by $\frac{\partial \psi}{\partial T} = 0$, we will have that $H_0\psi = 0$ and due to the algebra (74) we obtain $S\psi = \bar{S}\psi = 0$ and the wave function ψ_* becomes ψ.

Acknowledgments. We are grateful to E. Ivanov, S. Krivonos, J.L. Lucio, I. Lyanzuridi, L. Marsheva, O. Obregón, M.P. Ryan, J. Socorro and M. Tsulaia for their interest in the work and useful comments. This research was supported in part by CONACyT under the grant 28454E. Work of A.I.P. was supported in part by INTAS grant 96-0538 and by the Russian Foundation of Basic Research, grant 99-02-18417. One of use J.J.R. would like to thank CONACyT for support under Estancias Posdoctorales en el Extranjero and Instituto de Física de la Universidad de Gto. for its hospitality during the final stages of this work.

---

[1] C.J. Isham and K. Kuchar, Ann. Phys. (N.Y.) **104** (1985) 316.

[2] B. DeWitt, Phys. Rev. **160** (1967) 1113.

[3] C.W. Misner, K.S. Thorne and J.J. Wheeler, *Gravitation* (W.M. Freedman, San Francisco) (1970).

[4] M. Henneaux and C. Teitelboim, *Quantization of Gauge sytems* (Princeton Univ. Press, Princeton, N.J.) (1992).

[5] D.M. Gitman and I.V. Tyutin, *Quantization of Fields with Constraints* (Springer Verlag, Berlin-Heidelberg-New-York) (1990).

[6] G. Fülöp, D.M. Gitman and I.V. Tyutin, Int. Jour. of Theor. Phys. **38** (1999) 1941.

[7] C.W. Misner, Phys. Rev. **186** (1969) 1319.

[8] M.P. Ryan, *Hamiltonian Cosmology* (Lectures Notes in Physics N_0 **13**, Springer Verlag, Berlin-Heidelberg-New-York) (1972).

[9] *"Quatntum Cosmology and Baby Universes"*, vol.**7**, edited by S. Coleman, J.B. Hartle, T. Piran and S. Weinberg. World Scientific (1991).

[10] C. Teitelboim, Phys. Rev. Lett. **38** (1977) 1106.

[11] A. Macías, O. Obregón and M.P. Ryan, Class. and Quantum Grav. **11** (1987) 1477.

[12] P.D. D'Eath and D.I. Hughes, Phys. Lett. **B 214** (1988) 498.

[13] P. Hajiacek and K. Kuchar, Phys. Rev. **D 41** (1990) 1091.

[14] K. Kuchar, Phys. Rev. **D 43** (1991) 3332.

[15] M. Gavaglia, V. De Alfaro and A.T. Filipov, Int. Jour. of Mod. Phys. **A 10** (1995) 611.

[16] V.G. Lapchinskii and V.A. Rubakov, Theor. Mat. Fiz. **33** (1977) 364.

[17] V. Pervushin, V. Papoyan, S. Gogilidze, A. Khvedelidze, Yu. Palii and V. Smirichinsky, Phys. Lett. **B365** (1996) 35.

[18] T. Banks, Nucl. Phys. **B 249** (1985) 332.

[19] J.J. Halliwell, Phys. Rev. **D36** (1987) 3626.

[20] P.D. D'Eath, *Supersymmetric Quantum Cosmology*, Cambridge University Press, (1996).
[21] L. Brink, P. DiVecchia and P. Howe, Nucl. Phys. **B118** (1977) 76.
[22] D.P. Sorokin, V.I. Tkach and D.V. Volkov, Mod. Phys. Lett. **A4** (1989) 901.
[23] V.I. Tkach, J.J. Rosales and O. Obregón, Class. and Quantum Grav. **13** (1996) 2349.
[24] O. Obregón, J.J. Rosales, J. Socorro and V.I. Tkach, Class. and Quantum Grav. **16** (1999) 2861.
[25] V.I. Tkach, J.J. Rosales and J. Socorro, Class. and Quantum Grav. **16** (1999) 797.
[26] V.I. Tkach, J.J. Rosales and J. Socorro, Mod. Phys. Lett. **A 14** (1999) 1209.